CHAPTER – 1

RELATIONS AND FUNCTIONS

FORMULAE :

(1) Cartesian Product :

A, B are any two non empty sets. Cartesian product of A and B

$= A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$. $A \times B \neq B \times A$.

If $n(A) = m$, $n(B) = n$ then $n(A \times B) = mn = n(B \times A)$ and

$n(A \times A) = m^2$.

(2) Domain and Range of Relation :

A, B are nay two sets. $R \subseteq A \times B \Rightarrow R$ is a relation from A to B.

$(x, y) \in R \Rightarrow x \in A$ and $y \in B$.

domain $R = \{x / (x, y) \in R\}$ and range $R = \{y / (x, y) \in R\}$,

$R \subseteq A \times A \Rightarrow R$ is a relation in A.

(3) Inverse Relation :

R is a relation from a set A to a set B. The set {(x, y) / (y, x) $\in$ R} is a relation from B to A. This relation is called the inverse relation of R and is denoted by R^{-1}.

Thus : $R \subseteq A \times B \Leftrightarrow R^{-1} \subseteq B \times A$ and $(y, x) \in R \Leftrightarrow (x, y) \in R^{-1}$

(4) Types of Relation :

(i) Reflexive : A relation R on a set A is said to be reflexive if for every $x \in A$, $(x, y) \in R$.

(ii) Symmetric : A relation R on a set A is said to be symmetric if $xRy \Rightarrow yRx$ i.e. $(x, y) \in R \Rightarrow (y, x) \in R$.

Note : The necessary and sufficient condition that relation R be symmetric $R = R^{-1}$.

(iii) Transitive : A relation R in set A is called transitive if xRy and $xRz \Rightarrow xRz$ i.e., xRy and $yRz \Rightarrow (x, z) \in R$.

(5) Equivalence relation :

A relation which is reflexive, symmetric and transitive is called an equivalence relation.

i.e., R is an equivalence relation in A if

(i) $(x, x) \in R$ for all $x \in A$ (ii) $(x, y) \in R \Rightarrow (y, x) \in R$

(iii) $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$

(6) Important results of Relation :

(a) A, B, C are sets. If $S \subseteq A \times B$, $R \subseteq B \times C$ and $RoS \subseteq A \times C$, then $(RoS)^{-1} = S^{-1}OR^{-1}$.

(b) If $f \subseteq A \times B$, $g \subseteq B \times C$, $h \subseteq C \times D$, then ho (gof) = (hog) of = hogo$f$.

(c) If R_1 and R_2 are two relations from A to B, then $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$ are also relations from A to B.

(d) (i) The inverse of an equivalence relation is an equivalence relation.

(ii) If R_1, R_2 are two equivalence relations on set A, then $R_1 \cap R_2$ is an equivalence relation whereas $R_1 \cap R_2$ is not necessarily an equivalence relation.

(7) Function : A relation f, which associates to each element of set A, a unique element of a set B, is called a function from A to B or A into B. We write A $\xrightarrow{f}$ B or f A $\rightarrow$ B.

We read as : f is a mapping from A to B into B. A is called domain of f and B is called co-domain of f.

(i) Every function from A to B is a relation from A to B. But every relation from A to B is not a function from A to B.

(ii) If f(a) = b for a $\in$ A and b $\in$ B, then b is called the f image of a. Also a is called the pre-image or inverse of b. b may have more than one pre-image or inverse image.

(iii) The set of all pre-image of b is denoted by f^{-1} (b) and f^{-1} (b) = {a / f (a) = b}.

(iv) For a set C $\subseteq$ B, the set of all x $\in$ A such that f(x) $\in$ C is called the inverse image set of C with respect to f and denoted by f^{-1} (C). i.e. f^{-1} (C) = {x $\in$ A / f (x) $\in$ C}.

(v) Range of f = f (A) = {f(x) / x $\in$ A}. Also f (A) $\subseteq$ B.

(vi) Every element of A has a unique f - image in B.

(vii) Two or more elements of A can have the same f image in B. There may be some elements in B which are not f - images of elements of A.

(8) Function as a set of ordered pairs.

$f \subseteq A \times B$. For all a $\in$ A if (a , x) $\in f$ for some x $\in$ B and (a, b) $\in f$, (a, c) $\in$

$f \Rightarrow$ b = c, then f is called a function from A to B.

(9) Equality of function :

Two functions f and g are said to be equal If

(i) They are defined on the same domain A and codomain B and

(ii) $f(x) = g(x)$ for every x $\in$ A. If for x $\in$ A, $f(x) \neq g(x)$, then

$f \neq g$.

(10) Number of function :

If n (A) = m, n(B) = n and $f : A \rightarrow B$, then number of possible functions (f) from A to B in n^m. Also n(A) $\gtrless$ n(B).

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(11) One – One function or Injection : f : A $\rightarrow$ B is a one-one function if a_1,

$a_2 \in$ A and $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$. Thus f : A $\rightarrow$ B is one-one if and only if for all $a_1, a_2 \in$ A, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. If $f(a_1) = f(a_2)$ in B does not imply $a_1 = a_2$, then f is not on-one.

A and B are finite sets and f : A $\rightarrow$ B is injection. Then

(i) n (A) $\leq$ n (B).

(ii) If n (A) = p and n(B) = q, then the number of all possible injection mappings is ${}^{q}P_{p}$.

(iii) n (A) = m and n(B) m $\Rightarrow$ possible number of one-one functions from A to B is m!

If y = f(x) is the graph drawn and if any line parallel to x-axis cuts the graph at only one point, then f(x) is one-one. Otherwise f(x) is not one-one.

f is a real function. f is one-one $\Rightarrow$ the graph of y = f(x) us either strictly increasing or strictly decreasing i.e., either $\frac{dy}{dx} = f'(x) > 0$ or < 0 for all x of dom f.

(12) Many – one mapping : If the mapping f : A $\rightarrow$ B is such that two distinct elements a_1, a_2 of A have the same f - image in B, then f is called a many–one mapping or many–one function.

(13) Onto function or Surjection : f : A $\rightarrow$ B is such that f (A) = B. Then f is an onto function from A to B.

Thus,

(i) there exists no element in B which is not the f image of some element of A.

(ii) A, B are finite sets $\Rightarrow$ n(A) $\geq$ n(B)

(iii) n(A) = m and n (B) = m $\Rightarrow$ possible number of onto functions from A to B is m!

(iv) n(A) = m and n(B) = 2 $\Rightarrow$ possible number of onto functions from A to B is $2^m - 2$.

(14) One – One onto function or Bijection : If the function is both one-one and onto, the function is called a bijection.

Thus,

(i) A, B are finite sets $\Rightarrow$ n(A) = n(B).

(ii) n(A) = m = n(B) $\Rightarrow$ possible number of bijections from A to B is m!

(15) Composite function or product function :

Let $f : A \to B$ and $g : B \to C$.

Let $a \in A$.

Since $f : A \to B$ there exists an element $b \in B$ such that $f(a) = b$.

Since $g : B \to C$ there exists an element $c \in C$ such that $g(b) = c \Rightarrow g(f(a)) = c \Rightarrow (gof)(a) = c$.

Thus gof is a function from A to C $\Rightarrow gof : A \to C$. gof is called composite mapping or product function of f and g.

We have

(i) codomain of $f \subseteq$ domain of g and range of $gof \subseteq$ range of g.

(ii) gof may be possible, but not fog.

(iii) Even if gof and fog are possible, fog may or may not be equal to gof.

(16) Some results of composite function :

(a) f : A $\to$ B and g : B $\to$ C are one-one $\Rightarrow$ go$f$: A $\to$ C is one-one. If f : A $\to$ B and g : B $\to$ C such that go$f$: A $\to$ C is one-one, then both f and g need not be one-one. But f is necessarily one-one.

(b) f : A $\to$ B and g : B $\to$ C are onto $\Rightarrow$ go$f$: A $\to$ C is onto. If f : A $\to$ B and g : B $\to$ C such that go$f$: A $\to$ C is onto, then f and g need not be onto. But g is necessarily onto.

(c) f : A $\to$ B and g : B $\to$ C are one-one onto $\Rightarrow$ go$f$: A $\to$ C is one-one onto. If f : A $\to$ B and g : B $\to$ C such that go$f$: A $\to$ C is one-one onto, then f is necessarily one-one and g is necessarily onto.

(d) If f : A $\to$ B, g : B $\to$ C and h : C $\to$ D, then ho (go$f$) = (hog) of i.e., composition of mappings is associative.

(17) Identity function : f : A $\rightarrow$ A defined by f(x) = x is called an identity function. It is denoted by I_A or I.

Every identity function is a bijection.

If f : A $\rightarrow$ B, then f o I_A = f and I_B of = f.

If f : A $\rightarrow$ a, Then fo I = Iof = f.

(18) Inverse function : f : A $\rightarrow$ B is a bijection. The relation f^{-1} is from B to A and is a unique bijection function. It is called the inverse function f.

We say that f is a one-one correspondence between A and B.

Only bijective function have inverses. The graph of a function and its inverse cue symmetrical about its line y = x.

(19) Some results of Identity and inverse function :

(a) If f : A $\rightarrow$ B is a bijection, then f^{-1} of = I_A and fof^{-1} = I_B. Also I_A^{-1} = I_A.

(b) If f : A $\rightarrow$ B and g : B $\rightarrow$ A are two functions such that go$f$ = I_A and $f$og = I_B, then g = f^{-1}.

(c) If f : A $\rightarrow$ B and g : B $\rightarrow$ C are bijections, then (go$f$)$^{-1}$ = f^{-1}o g^{-1}.

(20) Constant function : The function f : A $\rightarrow$ B is called a constant function, if the range of f consists of only one element i.e. for all x $\in$ A, f(x) = k where k is a fixed element of B.

Constant function is a bijection $\Rightarrow$ Domain and codomain of the function are singleton sets.

(21) Sum, Difference, Product and Quotient of Real Functions :

f is a real function with domain f = A and g is a real function with domain g = B.

Then we define:

(i) $(f + g)(x) = f(x) + g(x)$ $\longrightarrow$ Domain of f + g is A $\cap$ B.

(ii) $(f - g)(x) = f(x) - g(x)$ $\longrightarrow$ Domain of f - g is A $\cap$ B.

(iii) $(fg)(x) = f(x) \, . \, g(x)$ $\longrightarrow$ Domain of fg is A $\cap$ B.

(iv) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ $\longrightarrow$ Domain of $\dfrac{f}{g}$ is {x/x $\in$ A $\cap$ B,

(v) $(f + k)(x) = kf(x)$, k is constant, $\longrightarrow$ Domain of f + k is A.

(vi) $(kf)(x) = kf(x)$, k is constant, $\longrightarrow$ Domain of kf is A

(vii) $f^n(x) = (f(x))^n$. $\longrightarrow$ Domain of f^n is A.

(viii) $(\sqrt{f})(x) = \sqrt{f(x)}$ $\longrightarrow$ Domain of a subset of A i.e.

$A \cap \{x \mid f(x) \geq 0\}$

(ix) $|f|(x) = |f(x)|$ $\longrightarrow$ Domain of $|f|$ is A.

(22) Odd and Even Function :

f is a function and for every x of its domain

(i) $f(-x) = f(x) \Leftrightarrow f$ is even (ii) $f(-x) = -f(x) \Leftrightarrow f$ is odd

(iii) $f(-x) \neq f(x)$ or $-f(x) \Leftrightarrow f$ is neither even nor odd.

(23) Exponential functions : $f(x) = a^x$ $(a > 0)$ is called an exponential function.

Domain of f = R, Range of f = R^+.

(24) Logarithmic function : $f(x) = \log_e x$ where $a \neq 1$ and a and x are positive numbers is called a 'logarithmic function'.

(25) Binary Operation : Let A be a non-empty set. A function from A to A is called unary operation on A.

Let A be a non-empty set. A function from $A \times A$ to A is called a binary operation on A. If a binary operation on A is denoted by *, then the image * (a, a') of $(a, a') \in A \times A$ under the binary operation * is generally written as a * a'.

Thus, a binary operation '*' on A is a rule which assign to a pair $a, a' \in A$ another unique element $a * a' \in A$.

(26) Types of Binary operations : Let A be a non-empty set and * be a binary operation on A.

(i) The binary operation * is said to be commutative if

$a * b = b * a$ for $a, b \in A$.

(ii) The binary operation * is said to be associative if

$(a * b) * c = a * (b * c)$ for $a, b\ c \in A$

(iii) The binary operation * is said to be left distributive over another binary operation o on A if

$a * (b \text{ o } c) = (a * b) \text{ o } (a * c)$ for $a, b, c \in A$

(iv) The binary operation * is said to be right distributive over another binary operation o on A if

$(b \text{ o } c) * a = (b * a) \text{ o } (c * a)$ for $a, b, c \in A$.

(27) Identity element and inverse of an elements :

Let A be a non-empty set and * be a binary operation of A.

(i) An element $e \in A$ is said to be an identity element for the binary operation * if

$a * e = a = e * a$ for $a \in A$

(ii) Fro $a \in A$, element $b \in A$ is said to be an inverse of if

$a * b = e = b * a$.

Remark. The inverse of an element is generally denoted by a^{-1}.

(28) Some results of Binary Operation :

(a) Let * be a binary operation defined on a non-empty set A. If identity element for * exists, then it is unique.

(b) For an associative binary operation * on A with identity, an invertible element possesses unique inverse.

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